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RESEARCH ARTICLE

Designs Robust Against Presence of an Outlier in an Analysis of Covariance Model

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Abstract
Presence of one or more aberration in the observations affects inference procedure in statistical analysis. Block designs robust against presence of aberrations can be found in the literature for both regression and block design set-ups. In this paper, an attempt has been made to find robust designs in a block design set-up with covariates, when there is one single wild observation in the study variable. Specifically, effects on the estimation of a full set of orthonormal treatment contrasts and that on the estimation of covariate parameters have been considered.

Keywords Block design · Covariates · Outlier · Robust design

Mathematics Subject Classification 62K25

1 Introduction

An outlier in a set of data is an observation which is inconsistent with the remainder of the observations in that data set. Outliers may occur in the data generated from experimental designs due to disease and/or insect attack on some particular plot of the experiment, heavy irrigation by mistake on some particular block(s) or plot(s) of the experiment, mistakes creeping in during recording of data, etc. In the context of

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response surface design, Box and Draper (1975) proposed a set of desirable properties for a design to be satisfactory. In the context of classical Gauss–Markov model $\left(\mathbf{Y}^{n\times 1}, \mathbf{X}^{n\times p}\boldsymbol{\theta}^{p\times 1}, \sigma^2\mathbf{I}_n\right)$, $\text{rank}(\mathbf{X}) = p$, they showed that in order that the predicted response vector is insensitive to the presence of an outlier, the quantity $\sum_{u=1}^{n} r_{uu}^2 = \delta$, say should be a minimum, where r_{uu} is the uth diagonal element of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. A design for which δ is minimum is called robust. In the context of block designs, primary interest is in the estimation of treatment contrasts. Mandal (1989) initiated the problem of finding robust designs for estimating a *full set* of orthonormal treatment contrasts and established the robustness of randomised block design (RBD) and balanced incomplete block design (BIBD). The problem was investigated for different set ups by a number of authors (e.g. Gopalan and Dey 1976; Ghosh and Roy 1982; Mandal and Shah 1993; Sarkar et al. 2003; Biswas 2012; Biswas et al. 2015).

In many experiments, besides the response under study, there may be one or more covariates which play an important role in the outcome of an experiment. Standard literature is available for the analysis of such experiment with covariates (see eg. Rao 1973). Harville (1975) considered the problem of choosing designs for the estimation of treatment contrasts when the covariate values are given. Lopes Troya (1982a, b) first examined the problem of optimum choice of controllable covariate values for the estimation of covariate parameters in the completely randomised design (CRD) set up. Afterwards, substantial contribution for the above problem in different block design set ups, is made in Das et al. (2003) and Dutta et al. (2009, 2010a) among others. The details of the choice of controllable covariates can be found in a recent book by Das et al. (2015). Now, if there are one or more outlying observations, either in the study variable or in the covariates, these will affect the inference process both for the design parameters as well as for the covariate parameters. However, there is scope of properly designing the experiment as well as judicious choice of the controllable covariate values to minimize the disturbance caused to the inference process by the outlying observation(s).

In this paper, we will be mainly concerned with the problem of estimating (1) a full set of orthonormal treatment contrasts and (2) the covariate parameters when there is only one outlying observation in the study variable.

In Sect. 2, the problem is formulated. In Sect. 3, the problem of finding robust designs for the estimation of a full set of orthonormal treatment contrasts is considered. In Sect. 4, covariate designs robust for the estimation of covariate parameters is examined. Finally, in Sect. 5, a discussion on the findings is given.

2 The Problem and the Perspective

Consider the usual analysis of covariance set up

$$(\mathbf{Y}, \mathbf{X}\boldsymbol{\theta} + \mathbf{Z}\boldsymbol{\gamma}, \sigma^2\mathbf{I}_n) \tag{2.1}$$

where $\mathbf{Y}^{\,n\times 1}$ is the response vector; $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ stand for the vectors of effects and covariate parameters respectively; $\mathbf{X}^{\,n\times p}$ and $\mathbf{Z}^{\,n\times q}$ represent design matrices corresponding to analysis of variance (ANOVA) part and the covariate part respectively; $\mathbf{I}_n$ is identity matrix of order n. We assume further that rank($\mathbf{X}$) = $r \leq \min\{p, n\}$ and rank$(\mathbf{X}, \mathbf{Z}) = r + q$ so that all γ-parameters are estimable (cf. Rao 1973). Suppose now that there is an aberration 'a' present in the uth observation y_u, making it an outlier. The standard least squares estimates ($\hat{\boldsymbol{\theta}}$, $\hat{\gamma}$) of the parameters ($\boldsymbol{\theta}$, γ) satisfy the following normal equations:

$$\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\theta}} + \mathbf{X}'\mathbf{Z}\hat{\gamma} = \mathbf{X}'\mathbf{y}$$
$$\mathbf{Z}'\mathbf{X}\hat{\boldsymbol{\theta}} + \mathbf{Z}'\mathbf{Z}\hat{\gamma} = \mathbf{Z}'\mathbf{y}$$

which gives

$$\hat{\gamma} = (\mathbf{Z}'\mathbf{Q}_{\mathbf{x}}\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{Q}_{\mathbf{x}}\mathbf{y}) = \mathbf{R}^{-1}(\mathbf{Z}'\mathbf{Q}_{\mathbf{x}}\mathbf{y}), \tag{2.2}$$

where

$$\mathbf{Q}_{\mathbf{x}} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}', \; \mathbf{R} = \mathbf{Z}'\mathbf{Q}_{\mathbf{x}}\mathbf{Z}. \tag{2.3}$$

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= \boldsymbol{\theta}_0 - \hat{\gamma}_1\boldsymbol{\theta}_1 - \hat{\gamma}_2\boldsymbol{\theta}_2 - \ldots - \hat{\gamma}_q\boldsymbol{\theta}_q, \\ \boldsymbol{\theta}_0 &= (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y}, \; \boldsymbol{\theta}_i = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{z}_i \end{aligned} \tag{2.4}$$

$\mathbf{X}$ is the design matrix for the ANOVA part and $\mathbf{Z} = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_q)$ is the design matrix for the covariate part, $\mathbf{z}_i$ is the $n \times 1$ vector of covariate values corresponding to the covariate i, $i = 1, 2, \ldots, q$. Here it is assumed that the covariate values are controllable i.e. it is assumed that the ith variable z_i varies in the domain [a_i, b_i] which by a change of origin and scale, can be reduced to $[-1, +1]$. Thus, without loss of generality, we assume that each of the q covariates varies in $[-1, +1]$ (cf. Das et al. 2015).

3 Robust Design for Treatment Contrasts

Let us consider the model:

$$(\mathbf{y}, \; \mathbf{D}_1\boldsymbol{\beta} + \mathbf{D}_2\boldsymbol{\tau} + \mathbf{Z}\gamma, \; \sigma^2\mathbf{I}) \tag{3.1}$$

where $\mathbf{y}$ is the response vector, $\mathbf{D}_1$ and $\mathbf{D}_2$ represent respectively the observation vs. block and observation vs. treatment incidence matrices, $\mathbf{Z}$ is the matrix of covariate values. $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$ represent respectively the vector of block effects together with the common mean and the vector of treatment effects. γ represents vector of covariate parameters and σ^2 is the error variance (cf. Equation (2.1)). In this section, we investigate with respect to Model (3.1), the problem of finding block designs robust against the presence of outliers for the estimation of a full set of orthonormal treatment contrasts, when one or more covariates are present in the model. Comparing Models (2.1) and (3.1), it is seen that for a block design set-up, $\mathbf{X}$ and $\boldsymbol{\theta}$ are given by

$$\mathbf{X} = (\mathbf{D}_1, \mathbf{D}_2), \boldsymbol{\theta} = (\boldsymbol{\beta}', \boldsymbol{\tau}')'. \tag{3.2}$$

respectively.

From (2.4), it follows that

$$\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}_0 - \gamma_1 \hat{\boldsymbol{\tau}}_1 - \gamma_2 \hat{\boldsymbol{\tau}}_2 - \ldots - \gamma_q \hat{\boldsymbol{\tau}}_q \tag{3.3}$$

where

$$\mathbf{C}\boldsymbol{\tau}_0 = \mathbf{G}\mathbf{y},\ \mathbf{C}\boldsymbol{\tau}_i = \mathbf{G}\mathbf{z}_i,\ i = 1, 2, \ldots, q;\ \mathbf{G} = \mathbf{D}_2' - \mathbf{N}'\mathbf{k}^{-\delta}\mathbf{D}_1',\ \mathbf{N} = \mathbf{D}_1'\mathbf{D}_2 \tag{3.4}$$

and $\mathbf{C}$ is the characteristic matrix of the ANOVA part given by $\mathbf{C} = \mathbf{r}^{\delta} - \mathbf{N}'\mathbf{k}^{-\delta}\mathbf{N}$ with $\mathbf{r}^{\delta} = \mathrm{diag}(r_1, r_2, \ldots, r_v)$ and $\mathbf{k}^{-\delta} = \mathrm{diag}(1/k_1, 1/k_2, \ldots, 1/k_b)$; r_j, k_i being the replication of the treatment j and the size of the block i respectively; $\mathbf{N}$ is the incidence matrix of the block design. Let $\mathbf{L}$ be an orthogonal matrix of order v and is of the form

$$\mathbf{L} = (\,(1/v)\mathbf{1}_v \quad \mathbf{P}\,).$$

Then $\mathbf{P}^{\,v \times v-1}$ satisfies

$$\mathbf{P}'\mathbf{1} = \mathbf{0}, \mathbf{P}'\mathbf{P} = \mathbf{I}_{v-1},\ \mathbf{P}\mathbf{P}' = \mathbf{I}_v - (1/v)\mathbf{1}_v\mathbf{1}_v'. \tag{3.5}$$

Suppose we are interested in inferring on a full set of orthonormal treatment contrasts

$$\boldsymbol{\phi} = \mathbf{P}'\boldsymbol{\tau}. \tag{3.6}$$

Without loss of generality, we can take

$$\mathbf{P} = [\xi_1, \xi_2, \ldots, \xi_{v-1}] \tag{3.7}$$

where $\xi_1, \xi_2, \ldots, \xi_{v-1}$ are orthonormal eigenvectors corresponding to the non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{v-1}$ of $\mathbf{C}$ matrix respectively. Then

$$\mathbf{P}'\mathbf{C}\mathbf{P} = \boldsymbol{\Lambda} = \mathrm{Diag}[\lambda_1, \lambda_2, \ldots, \lambda_{v-1}] \tag{3.8}$$

and the best linear unbiased estimator (BLUE) of $\boldsymbol{\phi}$ is given by $\hat{\boldsymbol{\phi}} = \mathbf{P}'\hat{\boldsymbol{\tau}}$ where $\hat{\boldsymbol{\tau}}$ is given by (3.3). It can be shown that

$$\mathbf{C}\hat{\boldsymbol{\tau}} = \mathbf{G}^{*}\mathbf{y} \tag{3.9}$$

where

$$\begin{aligned} \mathbf{G}^{*} &= \mathbf{G}\left[\mathbf{I} - (\mathbf{Z}\mathbf{R}^{-1}\mathbf{Z}')\mathbf{Q}_X\right] \\ &= \mathbf{G} - \mathbf{G}\mathbf{Z}\mathbf{R}^{-1}\mathbf{Z}'\mathbf{Q}_X \end{aligned} \tag{3.10}$$

and $\mathbf{G}$ is given by (3.4).

From (3.9), we have

$$(\mathbf{P}'\mathbf{C}\mathbf{P})(\mathbf{P}'\hat{\boldsymbol{\tau}}) = \mathbf{P}'\mathbf{G}^{*}\mathbf{y} \tag{3.11}$$

Let us restrict to the class of connected block designs so that all treatment contrasts are estimable. As a consequence, rank$(\mathbf{P}'\mathbf{C}\mathbf{P}) = \text{rank}(\mathbf{C}) = v - 1$ and that from (3.11) and (3.8) we have

$$\hat{\boldsymbol{\phi}} = \mathbf{P}'\hat{\boldsymbol{\tau}} = (\mathbf{P}'\mathbf{C}\mathbf{P})^{-1}\mathbf{P}'\mathbf{G}^*\mathbf{y} = \boldsymbol{\Lambda}^{-1}\mathbf{P}'\mathbf{G}^*\mathbf{y} \tag{3.12}$$

It is easy to see that

$$E(\hat{\boldsymbol{\phi}}) = \boldsymbol{\phi},\ Disp(\hat{\boldsymbol{\phi}}) = \sigma^2\mathbf{W} \tag{3.13}$$

where

$$\begin{aligned} \mathbf{W} &= \boldsymbol{\Lambda}^{-1}\mathbf{P}'\mathbf{G}[\mathbf{I} - (\mathbf{Z}\mathbf{R}^{-1}\mathbf{Z}')\mathbf{Q}_X]\mathbf{G}'\mathbf{P}\boldsymbol{\Lambda}^{-1} \\ &= \boldsymbol{\Lambda}^{-1}\mathbf{P}'\mathbf{G}\mathbf{G}'\mathbf{P}\boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1}\mathbf{P}'\mathbf{G}(\mathbf{Z}\mathbf{R}^{-1}\mathbf{Z}')\mathbf{Q}_X\mathbf{G}'\mathbf{P}\boldsymbol{\Lambda}^{-1} \\ &= \boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1}\mathbf{P}'\mathbf{G}\mathbf{Z}\mathbf{R}^{-1}\mathbf{Z}'\mathbf{G}'\mathbf{P}\boldsymbol{\Lambda}^{-1}\ \ (\text{Since}\ \mathbf{G}\mathbf{D}_1 = \mathbf{0}) \end{aligned} \tag{3.14}$$

Now if there is an aberration in the uth observation of the study variable y, then from (3.12) the discrepancy in the ith component of $\hat{\boldsymbol{\phi}}$ will be t_{iu} where t_{iu} is the (i, u)th element of $\mathbf{T}$ where

$$\mathbf{T} = (\mathbf{P}'\mathbf{C}\mathbf{P})^{-1}\mathbf{P}'\mathbf{G}^* = \boldsymbol{\Lambda}^{-1}\mathbf{P}'\mathbf{G}^*. \tag{3.15}$$

We can now define a measure of the overall impact of the aberration in the uth observation on $\hat{\boldsymbol{\phi}}$ as (cf. Mandal 1989)

$$t_u^* = \mathbf{t}_u'\mathbf{W}^{-1}\mathbf{t}_u = (\mathbf{T}'\mathbf{W}^{-1}\mathbf{T})_{u,u} \tag{3.16}$$

where $\mathbf{t}_u = (t_{1u}, t_{2u}, \ldots, t_{v-1,u})'$. For a robust design t_u^*s, given by (3.16), should be as equal as possible (cf. Mandal 1989; Mandal and Shah 1993).

Now if $\mathbf{G}\mathbf{Z} = \mathbf{D}_2'\mathbf{Z} - \mathbf{N}'\mathbf{k}^{-\delta}\mathbf{D}_1'\mathbf{Z} = \mathbf{0}$ then from (3.10), $\mathbf{G}^* = \mathbf{G}$ and from (3.14), $\mathbf{W} = \boldsymbol{\Lambda}^{-1}$ and hence condition (3.16) reduces to

$$t_u^* = (\boldsymbol{\Lambda}^{-1}\mathbf{G}\boldsymbol{\Lambda}\mathbf{G}'\boldsymbol{\Lambda}^{-1})_{u,u} \tag{3.17}$$

which does not involve $\mathbf{Z}$. Hence the problem reduces to that of finding a block design robust against outliers when $\mathbf{Z}$ satisfies $\mathbf{G}\mathbf{Z} = \mathbf{0}$. From (3.17) it is seen that if the design (without covariate) is connected and variance balanced (i.e. $\lambda_1 = \lambda_2 = \ldots = \lambda_{v-1}$) then the design is always robust design against presence of single outlier provided $\mathbf{G}\mathbf{Z} = \mathbf{0}$. Now we can state this in the form of following theorem.

Theorem 3.1 *For the estimation of a full set of orthonormal treatment contrasts* $\boldsymbol{\phi}$, *a variance balanced block design remains robust if covariates are included in the model provided* $\mathbf{G}\mathbf{Z} = \mathbf{0}$.

Since RBD and BIBD are connected variance balanced designs, they are robust against presence of single outlier for the estimation of $\boldsymbol{\phi}$ in a block design model. Therefore the corollary follows:

Corollary 3.1 *In a block design set-up with covariates (3.1), an RBD and a BIBD remain robust against presence of an aberration, for the estimation of a full set of orthonormal treatment contrasts, provided* $\mathbf{GZ} = \mathbf{0}$.

Example 3.1 Consider an RBD with $b = 6,\ v = 4,$ where $\mathbf{D}_1 = (\mathbf{I}_6 \otimes \mathbf{1}_4),\ \mathbf{D}_2 = (\mathbf{1}_6 \otimes \mathbf{I}_4), \mathbf{N} = \mathbf{1}_6\mathbf{1}_4', \mathbf{k}^{-\delta} = \frac{1}{4}\mathbf{I}_6$ where $\mathbf{I}_t$ is an identity matrix of order t and $\mathbf{1}_t$ is a vector of all elements unity of order t.

Now $\mathbf{Z}$ is constructed by taking any q columns $(1 \le q < 10)$ from

$$\mathbf{U}^{24\times 12} = m\mathbf{H}_{12}^{**} \otimes \mathbf{H}_2^{*};\ -1 \le m \le 1$$

where $\mathbf{H}_{12} =$ Hadamard matrix of order 12

$$= \begin{pmatrix} 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix} = (\mathbf{H}_{12}^{**}, \mathbf{h}_{11}, \mathbf{1}),$$

$$\mathbf{H}_2 = \text{Hadamard matrix of order } 2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = (\mathbf{H}_2^{*}, \mathbf{1}),$$

and $\otimes$ denotes Kronecker product of matrices (cf. Rao 1973).

We can easily check that $\mathbf{D}_1'\mathbf{Z} = 0,\ \mathbf{D}_2'\mathbf{Z} = 0$ and then $\mathbf{GZ} = \mathbf{0}$.

However it is difficult to find robust design in the general set up achieving $\mathbf{GZ} = \mathbf{0}$. But for the class of connected, equireplicate, proper block design set up, we can formulate some sufficient conditions:

$$\mathbf{D}_1'\mathbf{Z} = c\mathbf{J}^{b\times q},\ \mathbf{D}_2'\mathbf{Z} = c\mathbf{J}^{v\times q};\ 0 \le |c| < \min\{k, r\} \tag{3.18}$$

then $\mathbf{GZ} = \mathbf{D}_2'\mathbf{Z} - \mathbf{N}'\mathbf{k}^{-\delta}\mathbf{D}_1'\mathbf{Z} = \mathbf{0}$.

Now we can state this in the form of following theorem.

Theorem 3.2 *For the estimation of a full set of orthonormal treatment contrasts* ϕ, *a connected, equireplicate, proper robust block design remains robust if covariates are included in the model provided condition (3.18) holds.*

Example 3.2 Consider a Symmetric Balanced Incomplete Block Design (SBIBD) with parameters $v = b = 7, r = k = 3, \lambda = 1$. This is constructed by developing the initial block (1, 2, 4) mod 7 and the seven blocks are (1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 0), (5, 6, 1), (6, 0, 2), (0, 1, 3) and the incidence matrix is given by

$$\mathbf{N} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now Hadamard matrix of order 4 can be written as

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{1}' \\ \mathbf{1} & \mathbf{H} \end{pmatrix} \tag{3.19}$$

where

$$\mathbf{H} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix} = (\mathbf{h}^{(1)}, \mathbf{h}^{(2)}, \mathbf{h}^{(3)}). \tag{3.20}$$

Now we consider the first row of incidence matrix i.e. (0,1,1,0,1,0,0) and replace the non zero positions of this row by the elements of $\mathbf{h}^{(1)}$ successively and develop the other rows by cyclically permuting it and we get

$$\mathbf{U}_1 = \begin{pmatrix} 0 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then from $\mathbf{U}_1$, we get $\mathbf{Z}^{(1)} = (-1, 1, -1, -1, -1, 1, 1, -1, -1, 1, -1, -1, -1, 1, -1, -1, 1, -1, -1, 1, -1)'$.

Similarly using $\mathbf{h}^{(2)}$ and $\mathbf{h}^{(3)}$, we get

$$\mathbf{Z}^{(2)} = (1, -1, -1, -1, 1, -1, -1, -1, 1, -1, -1, 1, 1, -1, -1, 1, -1, -1, 1, -1, -1)'.$$
$$\mathbf{Z}^{(3)} = (-1, -1, 1, 1, -1, -1, -1, 1, -1, -1, 1, -1, -1, -1, 1, -1, -1, 1, -1, -1, 1)'$$

respectively.

Now if we take $\mathbf{Z} = (\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}, \mathbf{Z}^{(3)})$ then it is seen that $\mathbf{D}_1'\mathbf{Z} = -\mathbf{J}^{b \times q}$, $\mathbf{D}_2'\mathbf{Z} = -\mathbf{J}^{v \times q}$ and so $\mathbf{GZ} = \mathbf{0}$ and the SBIBD is robust.

Remark 3.1 It is seen that if $c = 0$ then $\mathbf{X}'\mathbf{Z}=\mathbf{0}$. Several designs robust against presence of outliers are there for other set-ups viz. partially balanced incomplete block design, treatment-control design etc. (see Biswas 2012). These designs remain robust for models with covariates provided $\mathbf{X}'\mathbf{Z} = \mathbf{0}$. Such covariate designs satisfying the above condition in different block design set-ups can be found in Das et al. (2015). In addition, if $\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_q$ is satisfied then these covariate designs are also globally optimum (cf. Das et al. 2015).

Example 3.3 As it is known that a Semi-Regular Group Divisible Design (SRGDD) is robust for estimation of a full set of orthonormal treatment contrasts (cf. Biswas 2012), we consider an SRGDD with parameters $v = b = 8, r = k = 4, m_1 = 4, \ n = 2, \lambda_1 = 0, \lambda_2 = 1$ which is obtained from OA(8, 7, 2, 2), denoted by $\mathbf{A}$ where:

$$\mathbf{A}' = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 2 & 3 & 6 & 7 & 10 & 11 & 14 \\ 1 & 4 & 6 & 7 & 9 & 12 & 14 \\ 2 & 4 & 5 & 7 & 10 & 12 & 13 \\ 1 & 3 & 5 & 8 & 10 & 12 & 14 \\ 2 & 3 & 6 & 8 & 9 & 12 & 13 \\ 1 & 4 & 6 & 8 & 10 & 11 & 13 \\ 2 & 4 & 5 & 8 & 9 & 11 & 14 \end{pmatrix} = (\mathbf{A}_1|\mathbf{A}_2).$$

Using $\mathbf{A}_1$, the SRGDD with above parameters is obtained and the incidence matrix $\mathbf{N}$ corresponding to the design is written as follows

$$\mathbf{N} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{H}_2$ and $\mathbf{H}_4$ are written as

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (\mathbf{h}_1, \mathbf{1}); \ \mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = (\mathbf{h}_1^*, \mathbf{h}_2^*, \mathbf{h}_3^*, \mathbf{1}).$$

Replacing the elements in the columns of $\mathbf{A}_2$ by those of $\mathbf{h}_1$, $\mathbf{A}_2^*(1)$ can be written as

$$\mathbf{A}_2^*(1) = \begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} = (\mathbf{a}_1^*(1), \mathbf{a}_2^*(1), \mathbf{a}_3^*(1)).$$

If the non-zero elements of each row of $\mathbf{N}$ are replaced by the four elements (±1) of first column $\mathbf{h}_1^*$ of $\mathbf{H}_4$ in that order, then $\mathbf{N}_1^*$ is obtained as

$$\mathbf{N}_1^* = \begin{array}{c} \text{Block} \\ \downarrow \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \text{Treatment} \rightarrow \\ \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Now using the Khatri–Rao product (see Rao 1973, page 30) between the first column $\mathbf{a}_1^*(1)$ of $\mathbf{A}_2^*(1)$ and $\mathbf{N}_1^*$, the $\mathbf{U}$-matrix can be constructed as

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \odot \mathbf{N}_1^* = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 \end{pmatrix} = \mathbf{U}'(1,1,1).$$

In this way, altogether 9 $\mathbf{U}$-matrices can be constructed for different choices of columns of $\mathbf{A}_2^*(1)$ and first three columns of $\mathbf{H}_4$. From these 9 $\mathbf{U}$ matrices we can get 9 columns of $\mathbf{Z}$ matrix by making a correspondence between the elements of the column of $\mathbf{Z}$ with the positive entries of $\mathbf{U}$'s (cf. Das et al. 2015). For example the first column of $\mathbf{Z}$ obtained by using $\mathbf{U}(1, 1, 1)$ is

$(1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,-1,1,-1,1,1,-1,1,-1,-1,$
$1,-1,1,1,-1,1,-1)'$

We can easily check that $\mathbf{X}'\ \mathbf{Z}=\mathbf{0}$ and $\mathbf{Z}'\mathbf{Z}=32\mathbf{I}_q;\ 1\le q\le 9$. So by Theorem 3.1 such designs are robust against abberation for the estimation of a full set of orthonormal treatment contrasts.

4 Robust Design for Covariate Parameters

In this section, we will first examine the impact of the presence of an outlier on the estimates of the γ parameters and then find the robust design $\mathbf{Z}$ when $\mathbf{X}$ is fixed. For this, it is reasonable to consider $\mathbf{X}$ corresponding to a robust block design when covariates are not present. Now, from (2.2) we have

$$\hat{\gamma}=\mathbf{H}\mathbf{y}, \tag{4.1}$$

where

$$\mathbf{H}=(\mathbf{Z}'\mathbf{Q}_{\mathbf{x}}\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Q}_{\mathbf{x}}. \tag{4.2}$$

It is well known that

$$E(\hat{\gamma})=\gamma,\ Disp(\hat{\gamma})=\sigma^2\mathbf{R}^{-1}, \tag{4.3}$$

where $\mathbf{R}$ is defined in (2.3).

If there is an aberration 'a' in the uth observation y_u, the discrepancy in the value of $\hat{\gamma}_i$ is given by $\delta_{iu}=ah_{iu}$ where h_{iu} is the (i,u)th element of the matrix $\mathbf{H}$, $i=1,2,\ldots,q;u=1,2,\ldots,n$. The overall discrepancy due to the aberration in the uth observation is taken as (cf. Mandal 1989)

$$d_u=\boldsymbol{\delta}_u'\mathbf{R}\boldsymbol{\delta}_u=a^2(\mathbf{H}'\mathbf{R}\mathbf{h})_{u,u} \tag{4.4}$$

where $\boldsymbol{\delta}_u=(\delta_{1u},\ \delta_{2u},\ldots,\delta_{qu})$, $(\mathbf{H}'\mathbf{R}\mathbf{h})_{u,u}$ is the uth diagonal element of $\mathbf{H}'\mathbf{R}\mathbf{h}$. Now since $\sum_{u=1}^{n}d_{u,u}$ is a constant, as before a design for which $d_{u,u}$s are as equal as possible will be robust.

Let us first fix $\mathbf{X}$ and then choose $\mathbf{Z}$ for which the above condition is satisfied.

However it is difficult to find robust design in the general set up. So we restrict our search to the connected, equireplicate, proper variance balanced design (VBD) set up to find robust design.

It is well known that for a connected VBD that

$\mathbf{C}=\theta_2\left(\mathbf{I}-\frac{1}{v}\mathbf{J}\right),\ \ \mathbf{C}^+=\theta_2^{-1}\left(\mathbf{I}-\frac{1}{v}\mathbf{J}\right),\ \ \mathbf{D}=\theta_1\left(\mathbf{I}-\frac{1}{b}\mathbf{J}\right),\ \ \mathbf{D}^+=\theta_1^{-1}\left(\mathbf{I}-\frac{1}{b}\mathbf{J}\right)$ for some non-negative θ_1 and θ_2; where $\mathbf{J}$ is a matrix whose all elements are 1, $\mathbf{D}$ is the information matrix for the block effects, $\mathbf{C}^+$ and $\mathbf{D}^+$ are Moore Penrose generalised inverses of $\mathbf{C}$ and $\mathbf{D}$ respectively.

Hence using $\mathbf{D}_1$ and $\mathbf{D}_2$ from (3.2)

$\hat{\tau} = \mathbf{C}^{+}\mathbf{Q} = \theta_2^{-1}(\mathbf{I} - \frac{1}{v}\mathbf{J})\mathbf{G}_2\mathbf{y} = \theta_2^{-1}(\mathbf{I} - \frac{1}{v}\mathbf{J})(\mathbf{D}_2' - \frac{1}{k}\mathbf{N}'\mathbf{D}_1')\mathbf{y}$, where $\mathbf{G}_2 = \mathbf{D}_2' - \frac{1}{k}\mathbf{N}'\mathbf{D}_1'$

$\hat{\beta} = \mathbf{D}^{+}\mathbf{P} = \theta_1^{-1}(\mathbf{I} - \frac{1}{b}\mathbf{J})\mathbf{G}_1\mathbf{y} = \theta_1^{-1}(\mathbf{I} - \frac{1}{b}\mathbf{J})(\mathbf{D}_1' - \frac{1}{r}\mathbf{N}\mathbf{D}_2')\mathbf{y}$. where $\mathbf{G}_1 = (\mathbf{D}_1' - \frac{1}{r}\mathbf{N}\mathbf{D}_2')$.

Thus,

$$(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' = \begin{pmatrix} \theta_1^{-1}\left(\mathbf{I} - \frac{1}{b}\mathbf{J}\right)\left(\mathbf{D}_1' - \frac{1}{r}\mathbf{N}\mathbf{D}_2'\right) \\ \theta_2^{-1}\left(\mathbf{I} - \frac{1}{v}\mathbf{J}\right)\left(\mathbf{D}_2' - \frac{1}{k}\mathbf{N}'\mathbf{D}_1'\right) \end{pmatrix} = \begin{pmatrix} \theta_1^{-1}\left(\mathbf{D}_1' - \frac{1}{r}\mathbf{N}\mathbf{D}_2'\right) \\ \theta_2^{-1}\left(\mathbf{D}_2' - \frac{1}{k}\mathbf{N}'\mathbf{D}_1'\right) \end{pmatrix}.$$

so that

$$\begin{aligned} \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' &= \theta_1^{-1}\mathbf{D}_1\left(\mathbf{D}_1' - \frac{1}{r}\mathbf{N}\mathbf{D}_2'\right) + \theta_2^{-1}\mathbf{D}_2\left(\mathbf{D}_2' - \frac{1}{k}\mathbf{N}'\mathbf{D}_1'\right) \\ &= \theta_1^{-1}\left(\mathbf{I}_b \otimes \mathbf{J}_k - \frac{1}{r}\mathbf{D}_1\mathbf{N}\mathbf{D}_2'\right) + \theta_2^{-1}\mathbf{D}_2\left(\mathbf{D}_2' - \frac{1}{k}\mathbf{N}'\mathbf{D}_1'\right). \end{aligned}$$

Now from (2.2) and (3.2)

$$\begin{aligned} \mathbf{R} &= \mathbf{Z}'\mathbf{Q}_{\mathbf{x}}\mathbf{Z} \\ &= \mathbf{Z}'\left(\mathbf{I} - \theta_1^{-1}\left(\mathbf{D}_1\mathbf{D}_1' - \frac{1}{r}\mathbf{D}_1\mathbf{N}\mathbf{D}_2'\right) - \theta_2^{-1}\mathbf{D}_2\left(\mathbf{D}_2' - \frac{1}{k}\mathbf{N}'\mathbf{D}_1'\right)\right)\mathbf{Z} \\ &= \mathbf{Z}'\mathbf{Z} - \theta_1^{-1}\left(\mathbf{Z}'\mathbf{D}_1\mathbf{D}_1'\mathbf{Z} - \frac{1}{r}\mathbf{Z}'\mathbf{D}_1\mathbf{N}\mathbf{D}_2'\mathbf{Z}\right) \\ &- \theta_2^{-1}\left(\mathbf{Z}'\mathbf{D}_2\mathbf{D}_2'\mathbf{Z} - \frac{1}{k}\mathbf{Z}'\mathbf{D}_2\mathbf{N}'\mathbf{D}_1'\mathbf{Z}\right) \end{aligned}$$

If $\mathbf{D}_1'\mathbf{Z} \propto \mathbf{J}^{b\times q}$, $\mathbf{D}_2'\mathbf{Z} \propto \mathbf{J}^{v\times q}$ and $\mathbf{Z}'\mathbf{Z}$ is symmetric completely then $\mathbf{R}$ is symmetric completely and hence $\mathbf{R}^{-1}$ is also symmetric completely. Also under these conditions, $(\mathbf{Q}_{\mathbf{x}}\mathbf{Z}\mathbf{Z}'\mathbf{Q}_{\mathbf{x}})_{u,u} = \text{constant}\ \forall u$ and hence $(\mathbf{H}'\mathbf{R}\mathbf{h})_{u,u} = \text{constant}\ \forall u$. Thus we get the following theorem.

Theorem 4.1 *For a given connected, equireplicate, proper variance balanced design (VBD) with set up of (2.1), if there exists a* $\mathbf{Z}$ *such that* $\mathbf{D}_1'\mathbf{Z} \propto \mathbf{J}^{b\times q}$, $\mathbf{D}_2'\mathbf{Z} \propto \mathbf{J}^{v\times q}$ *and* $\mathbf{Z}'\mathbf{Z}$ *is completely symmetric, then it is robust against presence of an outlier in the response variable for estimation covariate parameters.*

Example 4.1 It is known that a BIBD is a connected, equireplicate, proper variance balanced design and we can easily check that the SBIBD(7,7,3,3,1) with initial block (1,2,4) along with $\mathbf{Z}$ given in Example 3.2 is robust since $\mathbf{R}$ is completely symmetric with $(\mathbf{H}'\mathbf{R}\mathbf{h})_{u,u} = \text{constant}\ \forall u$ as $\mathbf{D}_1'\mathbf{Z}$, $\mathbf{D}_2'\mathbf{Z}$ and $\mathbf{Z}'\mathbf{Z}$ satisfy the condition of Theorem 4.1. So the SBIBD of Example 3.2 is robust for the estimation of covariate parameters.

Remark 4.1 From Theorem 4.1, it is obvious that for a given robust connected, equireplicate, proper variance balanced design (VBD) set up if $\mathbf{Z}$ satisfies $\mathbf{D}_1'\mathbf{Z} = \mathbf{0}, \mathbf{D}_2'\mathbf{Z} = \mathbf{0}$ and $\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_q$ is still robust and this is also a globally optimal design for estimation of covariate parameters (cf. Das et al. 2015).

Example 4.2 Consider the RBD with $b = 6$, $v = 4$ in Example (3.1) as RBD is VBD. $\mathbf{Z}$ is constructed by taking any q $(1 \leq q < 10)$ columns from $\mathbf{H}_{12}^{**} \otimes \mathbf{H}_4^{*}$. We can easily check that $\mathbf{D}_1'\mathbf{Z} = \mathbf{0}$, $\mathbf{D}_2'\mathbf{Z} = \mathbf{0}$ and $\mathbf{Z}'\mathbf{Z} = 24\mathbf{I}_q$. So the covariate design in the above RBD is robust as well as globally optimal.

Corollary 4.1 *A connected variance balanced design is robust for joint estimation of* $\mathbf{P}'\boldsymbol{\theta}$ *and* γ *in corresponding covariate model if the design is robust for estimation of* $\mathbf{P}'\boldsymbol{\theta}$ and γ *separately in the corresponding covariate model.*

5 Discussion

In this paper, robust designs against presence of an outlier in the response variable have been investigated for the set-up of ANCOVA model. Both the estimation of a full set of orthonormal treatment contrasts and covariate parameters in a block design set-up have been considered for estimation of a complete set of orthonormal treatment contrasts as well as for the estimation of covariate parameters provided the covariate design satisfies certain conditions. Moreover, the robust designs, in most cases, found to be optimum with respect to different optimality criteria both for estimation of covariate parameters and for the estimation of a full set of orthonormal treatment contrasts. This can be extended to other set-ups viz. row-column design set-up, treatment-control design set-up etc.

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